Basic Computational Exercises in Game Theory

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Abstract

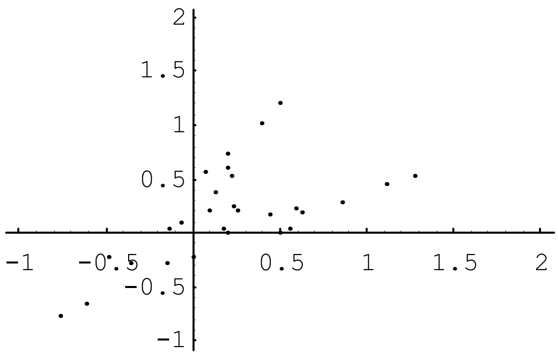
A computer algebra system (CAS) is used in this paper for developing the applications in game theory. Mathematica has been chosen as the programming tool. Simple algorithms are employed to find graphical solutions of 2×2 games. In contrast, Nash equilibria are found using a fixed point approach combined with a numerical optimization procedure.

1 Geometrical Representation of a 2×2 Game

We are interested in making a geometrical plot of the possible payoffs of a game. Along the horizontal axes we plot player 1's utility, and along the vertical, player 2's. Only certain combinations can arise; these are shown as the shaded area. To each pair of mixed strategies there corresponds a payoff which is one of the points in the shaded region; conversely, to each point in the region there corresponds at least one pair of strategies with this point as payoffs.

Exercise 1 *Luce and Raiffa (1957), page 93.*

	t_1	t_2
s_1	$(2,1)$	$(-1,-1)$
s_2	$(-1,-1)$	$(1,2)$



The Mathematica code that generate this graph is[1]:

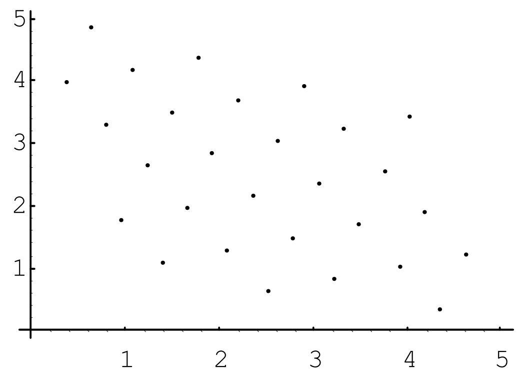
```
{{{a,b},{c,d}},{{e,f},{g,h}}}={{{2,1},{-1,-1}},{{-1,-1},{1,2}}};
value1= a*i*j + c*i*(1-j) + e*(1-i)*j + g*(1-i)*(1-j);
value2= b*i*j + d*i*(1-j) + f*(1-i)*j + h*(1-i)*(1-j);
Simplify[value1];
Simplify[value2];
tt=Table[ value1, {i , 0, 1, 0.05}, {j, 0, 1, 0.01}];
ss=Table[ value2, {i , 0, 1, 0.05}, {j, 0, 1, 0.01}];
uu=Flatten[tt];
vv=Flatten[ss];
ww=MapThread[List, {uu, vv}];
ListPlot[ww]
```

[1]The Mathematica version used in this paper is 4.0. I prefered to keep the programs as they were made long ago, when my knowledge of Mathematica was rather naive.

Exercise 2 *Gibbons (1992), page 96.*

	t_1	t_2
s_1	$(1,1)$	$(5,0)$
s_2	$(0,5)$	$(4,4)$

Similarly, we can modify the program in Exercise 1 to obtain the following graph:



2 Graphical Mixed Solutions of 2×2 Games

In this section we are interested in finding best response graphical solutions for simultaneous-move games of complete information. We use a graphical argument to show that any two-player game in which each player has two pure strategy has a Nash equilibrium, possible involving mixed strategies. In any game, a Nash equilibrium, involving pure or mixed strategies, appears as an intersection of the players' best-response correspondences (or functions). The subsections below are based on Gibbons (1992).

2.1 Best Response Functions

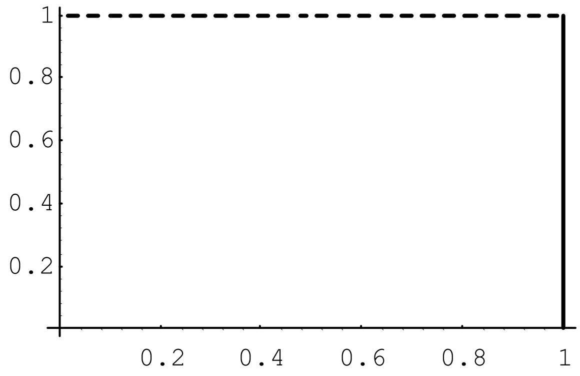
The simplest case we can find solving a 2×2 game occurs when there exists dominance between strategies. If $a > e$ and $c > g$, s_1 strictly dominates s_2. Accordingly s_3 strictly dominates s_4 when $b > d$ and $f > h$. Thus, player 1 will play strategy s_1 with probability 1, and player 2 will play strategy s_3 with probability 1. This is shown in Exercise 3.

		y	$1-y$
		s_3	s_4
x	s_1	(a, b)	(c, d)
$1-x$	s_2	(e, f)	(g, h)

Exercise 3

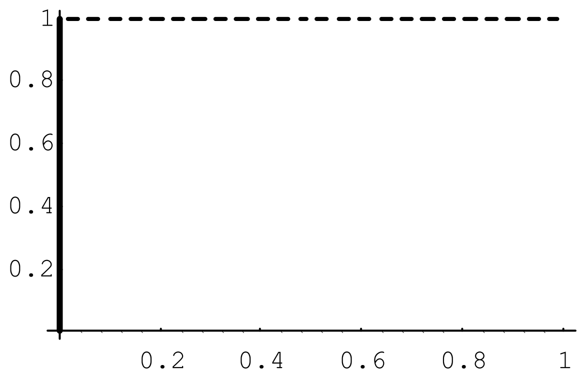
	s_3	s_4
s_1	$(3, 4)$	$(6, 2)$
s_2	$(2, 5)$	$(5, 3)$

The thick line corresponds to player 1 best reponse function, and the dashed line to the best response function of player 2.



Exercise 4

	s_3	s_4
s_1	$(2,4)$	$(5,2)$
s_2	$(3,5)$	$(6,3)$



In order to program a game in Mathematica it is necessary to define the game in matrix form:

```
game1={{{2,4},{5,2}},{{3,5},{6,3}}}
```

Then we can apply the Mathematica code for any case involving dominance:

```
{{{a,b},{c,d}},{{e,f},{g,h}}}=game1;
If[(a>e) && (c>g) && (b>d), (i=1) && (j=1)];
If[(a>e) && (c>g) && (b<d), (i=1) && (j=0)];
If[(a<e) && (c<g) && (f>h), (i=0) && (j=1)];
If[(a<e) && (c<g) && (b<d), (i=0) && (j=0)];
If[(b>d) && (f>h) && (a>e), (i=1) && (j=1)];
If[(b>d) && (f>h) && (a<e), (i=0) && (j=1)];
If[(b<d) && (f<h) && (c>g), (i=1) && (j=0)];
If[(b<d) && (f<h) && (c<g), (i=0) && (j=0)];
xx=Table[{i,p2},{p2,0,1,0.01}];
yy=Table[{p2,j},{p2,0,1,0.01}];
SetOptions[ListPlot, DisplayFunction->Identity];
x11=ListPlot[xx,PlotJoined->True, PlotStyle->Thickness[0.01]];
y12=ListPlot[yy,PlotJoined->True, PlotStyle->{Thickness[0.0095],
Dashing[{0.02}]}];
SetOptions[ListPlot, DisplayFunction->$DisplayFunction];
If[(i!=2) && (j!=2),
Show[x11,y12, DisplayFunction->$DisplayFunction]];
```

2.2 Best Response Correspondences

If there exists a value of x in which the best response graphs has more than one value, we will not have functions but correspondences.

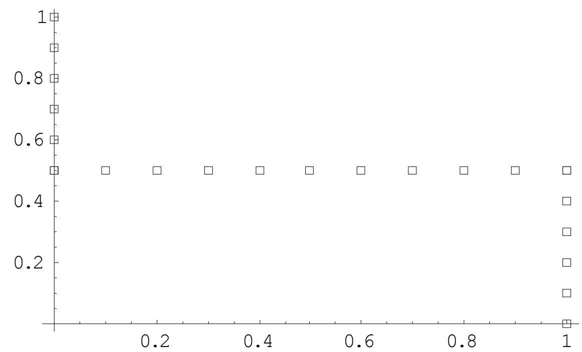
Exercise 5 *Binmore (1992), page 280.*

	s_3	s_4
s_1	$(1,-1)$	$(4,-4)$
s_2	$(3,-3)$	$(2,-2)$

The correspondence for player 1 is

$$C_1(y) = \begin{cases} \{1\} & if \quad 0 \leq y \leq \frac{1}{2} \\ [0,1] & if \quad y = \frac{1}{2} \\ \{0\} & if \quad \frac{1}{2} < y < 1 \end{cases}$$

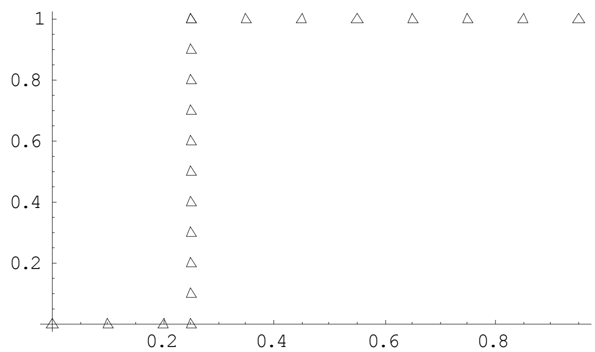
which in graphical terms becomes



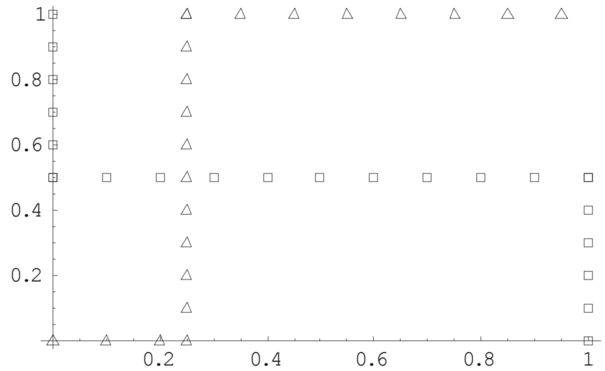
The correspondence for player 2 is

$$C_2(x) = \begin{cases} \{0\} & if \quad 0 \leq x \leq \frac{1}{4} \\ [0,1] & if \quad x = \frac{1}{4} \\ \{1\} & if \quad \frac{1}{4} < x < 1 \end{cases}$$

which in graphical terms is represented by



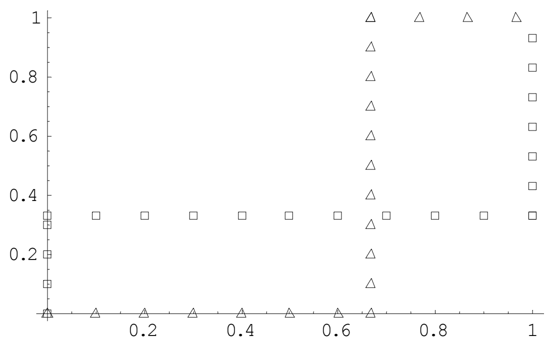
Thus, the equilibrium in this game appears as an intersection of the players' best-response:



Exercise 6

	s_3	s_4
s_1	$(2, 1)$	$(0, 0)$
s_2	$(0, 0)$	$(1, 2)$

The graphical solution is



where the square and triangle symbol correspond to player 1 and 2, respectively.

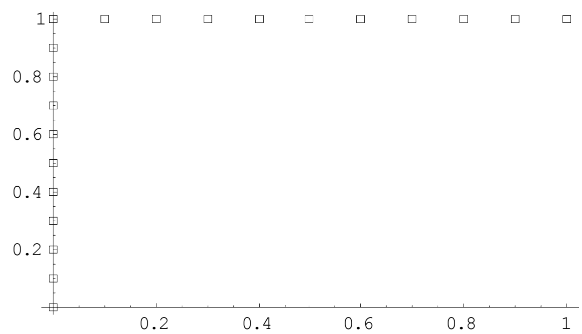
2.3 Best Response Correspondences with a Continuum of Mixed Equilibrium

If we modify the game in Exercise 6 we can illustrate the existence of a continuum of mixed equilibrium, where one player chooses a mixed strategy and the other player chooses a pure strategy.

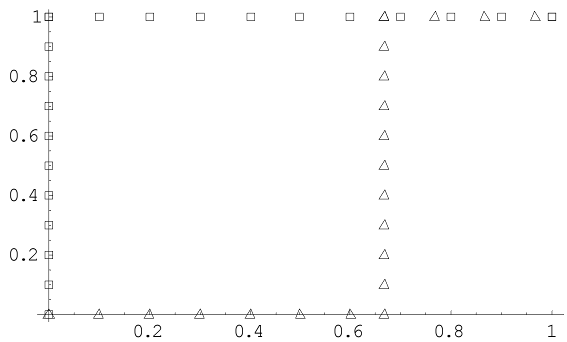
Exercise 7

	s_3	s_4
s_1	$(2,1)$	$(0,0)$
s_2	$(2,0)$	$(1,2)$

The best response for player 1 is now:



The equilibrium for this game involves a convex combination. Any time one player plays the same strategy in two equilibria, then any convex combination of these strategies is also an equilbrium:



To generate the figures in subsections 2.2 aand 2.3 we need to load the package MultipleListPlot as follows:

```
<< Graphics‘MultipleListPlot‘
```

To draw the graphs and solving the game, we set values in the ”If” line defined by the payoffs structure. The “If” statement in the first line follows the algorithm given in Gibbons (1992).

```
{{{a, b}, {c, d}}, {{e, f}, {g, h}}} = game1;
If[(a >= e) && (c <= g) && (b >= d) && (f <= h),
V[i_, {p1_, p2_}] := {p1, 1 - p1}.Transpose[game1, {2, 3,
1}][[i]].{p2,
1 - p2};
aa = V[1, {p1, p2}];
bb = V[2, {p1, p2}];
cc = Flatten[Solve[D[aa, p1] == 0, p2]];
dd = Flatten[Solve[D[bb, p2] == 0, p1]];
mixed2 = p2 /. cc;
mixed1 = p1 /. dd;
ee = p2/2 /. cc;
ff = p1/2 /. dd;
p2 = ee;
p1 = ff;
Clear[p1];
gg = aa /. p2 -> p2;
p1 = ff;
```

```
Clear[p2];
hh = bb /. p1 -> p1;
Clear[p1];
Clear[p2];
ii = D[gg, p1];
jj = D[hh, p2];
```

Testing the slope of functions "ii" and "jj" we can build up the values for the functions or correspondences that we need to draw.

```
If[ii < 0, pp1 = 0, pp1 = 1];
If[ii < 0, ppp1 = 1, ppp1 = 0];
If[jj < 0, pp2 = 0, pp2 = 1];
If[jj < 0, ppp2 = 1, ppp2 = 0];
oo = Table[{pp1, p2}, {p2, 0, mixed2, 0.1}];
pp = Table[{ppp1, p2}, {p2, mixed2, 1, 0.1}];
qq = Table[{p1, pp2}, {p1, 0, mixed1, 0.1}];
rr = Table[{p1, ppp2}, {p1, mixed1, 1, 0.1}];
ss = Table[{mixed1, p2}, {p2, 0, 1, 0.1}];
tt = Table[{p1, mixed2}, {p1, 0, 1, 0.1}];
```

Once we have the "values" we draw using the following code:

```
zz = MultipleListPlot[oo, pp, tt,
SymbolShape -> {PlotSymbol[Box, 2, Filled -> False]}];
ww = MultipleListPlot[qq, rr, ss,
SymbolShape -> {PlotSymbol[Triangle, 3, Filled -> False]}];
Show[zz, ww];
Show[GraphicsArray[{zz, ww, yy}]]]
```

3 Finding Nash equilibrium: The Lyapunov approach

The problem of finding a Nash equilibrium can be formulated as a problem of finding the minimum of a real valued function. In this approach, every isolated Nash equilbrium has a basin of attraction. Thus, if one starts close enough to an isolated Nash equilbrium, then one can guarantee to find it with any level of accuracy desired (McKelvey and McLennan (1996).

Judd (1998) proposes this methodology as an exercise for applying numerical optimization methods. Here, our interest is on game theory applications not on numerical procedures.

Let $M_i(s)$ be the payoff of player i corresponding to a strategy combination $s \equiv (s_1, s_2, ..., s_n) \in S$ where n is the number of players. S is the set of all possible strategy combinations. $\sigma(s)$ is the joint probability associated with s when the players play mixed strategies. Then, the payoff over the joint mixed strategies is

$$M_i(\sigma) = \sum_{s \in S} \sigma(s)\ M_i(s)$$

Let $M_i(s_{ij}, \sigma_{-i})$ the payoff of player i playing their jth pure strategy, while all other players play their components of $\sigma(s)$.

We can define the function

$$v(\sigma) = \sum_{i=1}^{n} \sum_{s_{ij} \in S_i} \{\max \left[M_i\left(s_{ij}, \sigma_{-i}\right) - M_i\left(\sigma\right), 0\right]\}^2$$

that is non-negative and is zero if and only if σ is a Nash equilibrium for the game. This function is everywhere differentiable. We want to minimize $v(\sigma)$ subject to the constraints that $\sum_j \sigma_{ij} = 1$ and $\sigma_{ij} \geq 0$.

The Lyapunov approach together with extensive search can therefore find all Nash equilibria if there are only finite number of equilibria. One disadvantage is that this method may get stuck at some local minimum not corresponding to any Nash equilibrium. In this case one have to check for $v(\sigma) = 0$.

The following exercises are suggested by Judd (1998).

Example 8 *Coordination game.*

		y s_3	$1-y$ s_4
x	s_1	$(1, 1)$	$(0, 0)$
$1-x$	s_2	$(0, 0)$	$(1, 1)$

In this game player 1 play her first strategy with probability x and player 2 her first strategy with probability y. First we separate the payoff "matrix" for player 1 and 2, i.e., $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and find $M_i(\sigma)$. The input code is:

```
payoffs={{1,0},{0,1}};
misigma={x,1-x}.payoffs.{y,1-y};
```

The terms $M_i(s_{ij}, \sigma_{-i})$ for player 1 are:

```
mplayer11={1,0}.payoffs.{y,1-y};
mplayer12={0,1}.payoffs.{y,1-y};
```

Thus, the first two terms for $v(\sigma)$ that corresponds to player 1 are:

```
player11=mplayer11-misigma;
player12=mplayer12-misigma;
```

Similarly, the corresponding terms for player 2 are:

```
mplayer21={x,1-x}.payoffs.{1,0};
mplayer22={x,1-x}.payoffs.{0,1};
player21=mplayer21-misigma;
player22=mplayer22-misigma;
```

The function to minimize $v(\sigma)$ is then:

```
vsigma=Max[0,player11]^2+Max[0,player12]^2+
Max[0,player21]^2+Max[0,player22]^2
```

Mathematica does not posses a built-in command for an optimization problem with constraint, so that, we should impose the constraints as penalty functions, as below:

```
g1=Max[0, x-1]^2;
g2=Max[0,-x]^2;
g3=Max[0,y-1]^2;
g4=Max[0,-y]^2;
```

The problem is solved with the FindMinimum procedure which need two starting points for each variable.

FindMinimum[vsigma+1000g1+1000g2+1000g3+1000g4, {x, 0.25,0.2}, {y,0.5,0.3}]

The outcomes are $x = 0.5$ and $y = 0.5$ and the function value is $1.97215 \times 10^{-31} \approx 0$.

To find a solution may sometimes require judicious choice of a starting point. We can try with:

FindMinimum[vsigma+1000g1+1000g2+1000g3+1000g4, {x, 0.75,0.85}, {y,0.93,0.97}]

The results in this case are $x = 1$ and $y = 1$ and the function value is $1.821141 \times 10^{-17} \approx 0$.

Using other starting points give rise to:

FindMinimum[vsigma+1000g1+1000g2+1000g3+1000g4, {x, 0.05,0.15}, {y,0.22,0.27}]

The results are $x = 4.43378 \times 10^{-8} \approx 0$ and $y = 3.85761 \times 10^{-11} \approx 0$ and the function value is $1.96584 \times 10^{-15} \approx 0$.

An example of a minimum that it is not a Nash equilibrium, can be obtained by:

FindMinimum[vsigma+10g1+10g2+10g3+10g4, {x, 0.1,0.2}, {y,0.25,0.3}]

The outcomes in this case are $x = 0.25$ and $y = 0.25$ and the function value is $0.03125 \neq 0$.

Thus, the three equilibria for this game are: {1,0,1,0}; {0.5,0.5,0.5,0.5};{0,1,0,1}.

Exercise 9

		x	q	$(1-p-q)$
		t_1	t_2	t_3
x	s_1	$(1,1)$	$(5,5)$	$(3,0)$
y	s_2	$(1,7)$	$(6,4)$	$(1,1)$
$(1-x-y)$	s_3	$(3,0)$	$(2,1)$	$(2,2)$

The Mathematica code for this game is:

```
payoff1={{1,5,3},{1,6,1},{3,2,2}};
payoff2={{1,5,0},{7,4,1},{0,1,2}};
player11={1,0,0}.payoff1.{p,q,(1-p-q)}-{x,y,(1-x-y)}.payoff1.{p,q,(1-p-q)};
player12={0,1,0}.payoff1.{p,q,(1-p-q)}-{x,y,(1-x-y)}.payoff1.{p,q,(1-p-q)};
player13={0,0,1}.payoff1.{p,q,(1-p-q)}-{x,y,(1-x-y)}.payoff1.{p,q,(1-p-q)};
player21={x,y,(1-x-y)}.payoff2.{1,0,0}-{x,y,(1-x-y)}.payoff2.{p,q,(1-p-q)};
player22={x,y,(1-x-y)}.payoff2.{0,1,0}-{x,y,(1-x-y)}.payoff2.{p,q,(1-p-q)};
player23={x,y,(1-x-y)}.payoff2.{0,0,1}-{x,y,(1-x-y)}.payoff2.{p,q,(1-p-q)};
const1=Max[0, x - 1]^2;
const2=Max[0,-x]^2;
const3=Max[0,y - 1]^2;
const4=Max[0,-y]^2;
const5=Max[0,x-y-1]^2;
const6=Max[0,p-q-1]^2;
const7=Max[0,p - 1]^2;
const8=Max[0,-p]^2;
const9=Max[0,q - 1]^2;
const10=Max[0,-q]^2;
vsigma= Max[0,player11]^2 + Max[0,player12]^2 + Max[0,player13]^2 + Max[0,player21]^2 + Max[0,player22]^2 + Max[0,player23]^2;
```

The game's solution is obtained using FindMinimum:

```
FindMinimum[vsigma+1000 const1+1000 const2+1000 const3+
1000 const4+1000 const5+ 1000 const6+1000 const7 +
1000 const8+1000 const9+1000 const10,
{x, 0.001, 0.02},{y, 0.25,0.26},
{p,0.52,0.55},{q,0.29,0.31}]
```

The results are: $x = -4.82642 \times 10^{-6} \approx 0$, $y = 0.250405$, $p = 0.538359$, $q = 0.307379$; and the value function is $1.15987 \times 10^{-6} \approx 0$.

A second solution is obtained with:

FindMinimum[vsigma+1000 const1+1000 const2+1000 const3+ 1000 const4+1000 const5+ 1000 const6+1000 const7 + 1000 const8+1000 const9+1000 const10, {x, 0.3, 0.03},{y, 0.3,0.2}, {p,0.3,0.2},{q,0.3,0.75}]

The outcomes are: $x = -4.98961 \times 10^{-9} \approx 0$, $y = 0.250402$, $p = 0.602625$, $q = 0.320525$; and the value function is $2.12732 \times 10^{-11} \approx 0$.

Therefore, the solutions of this game are:{{0, 0.25, 0.75},{0.538359,0.307379, 0.154262} and {{0, 0.25, 0.75}, {0.602665, 0.320525, 0.07}}

Exercise 10 *Fudenberg and Tirole (1991), page 55.*

		y	$1-y$
		L	R
x	U	$0,1,3$	$0,0,0$
$1-x$	D	$1,1,1$	$1,0,0$
		A	
		p	

	L	R
U	$2,2,2$	$0,0,0$
D	$2,2,0$	$2,2,2$
	B	
	q	

	L	R
U	$0,1,0$	$0,0,0$
D	$1,1,0$	$1,0,3$
	C	
	$1-p-q$	

First, we work out $M_i\left(s_{ij}, \sigma_{-i}\right)$ for player 1. Thus, the code is:

**m1=Simplify[{{x,1-x}.{{0,0},{1,1}}.{y,1-y}, {x,1-x}.{{2,0},{2,2}}.{y,1-y},
{x,1-x}.{{0,0},{1,1}}.{y,1-y}}.{p,q,1-p-q}];**

Similarly, for player 2 and 3:

**m2=Simplify[{{x,1-x}.{{1,0},{1,0}}.{y,1-y}, {x,1-x}.{{2,0},{2,2}}.{y,1-y},
{x,1-x}.{{1,0},{1,0}}.{y,1-y}}.{p,q,1-p-q}];
m3=Simplify[{{x,1-x}.{{3,0},{1,0}}.{y,1-y}, {x,1-x}.{{2,0},{0,2}}.{y,1-y},
{x,1-x}.{{0,0},{0,3}}.{y,1-y}}.{p,q,1-p-q}];**

The term $M_1(s_{ij}, \sigma_{-i})$ is obtained replacing $(x, 1-x)$ by $(1,0)$ in m1:

```
m11= m1 /. x->1;
```

Accordingly, all other terms are obtained in the same manner:

```
m12= m1 /. x->0;
m21= m2 /. y->1;
m22= m2 /. y->0;
m31= m3 /. {p->1, q->0};
m32= m3 /. {p->0, q->1};
m33= m3 /. {p->0, q->0};
```

Thus, $v(\sigma)$ and the constraints are equal to:

```
vsigma= Max[m11-m1,0]^2 + Max[m12-m1,0]^2 + Max[m21-
m2,0]^2 +
 Max[m22-m2,0]^2+ Max[m31-m3,0]^2 + Max[m32-m3,0]^2 +
 Max[m33-m3,0]^2;
const1=Max[0, x - 1]^2;
const2=Max[0,-x]^2;
const3=Max[0,y - 1]^2;
const4=Max[0,-y]^2;
const5=Max[0,p+q-1]^2;
const6=Max[0,p - 1]^2;
const7=Max[0,-p]^2;
const8=Max[0,q - 1]^2;
const9=Max[0,-q]^2;
```

Finally the minimization problem that we have to solve is:

```
FindMinimum[vsigma+1000 const1+1000 const2+1000 const3+
1000 const4+1000 const5+ 1000 const6+1000 const7 +
1000 const8+1000 const9,
 {x, 0.2, 0.6},{y, 0.2,0.6},
 {p,0.2,0.6},{q,0.2,0.6}]
```

which give us the following solution: $x = 0.000172152 \approx 0$, $y = 0.999848 \approx 1$, $p = 1$, $q = 0.000013552 \approx 0$; and the value function is $2.41631 \times 10^{-7} \approx 0$.

The unique equilibrium for this game is: {{0,1},{1,0},{1,0,0}}.

References

[1] Binmore, K. (1992). Fun and Games. D.C. Heath and Company.

[2] Fudenberg D. and J. Tirole (1991). Game Theory. MIT Press.

[3] Gibbons, R. (1992). A Primer in Game Theory. Harvester Wheatsheaf.

[4] Judd, K.L. (1998). Numerical Methods in Economics. MIT Press.

[5] Luce, R.D. and H.Raiffa (1957). Games and Decisions. Dover Publications.

[6] McKelvey, D and A. McLenan. (1996). Computation of Equilibria in Finite Games. in H.M Amman .D. Kendrick y J. Rust (eds) Handbook of Computational Economics, Vol I.